A new accurate scheme on polyhedral meshes for steady advection-reaction problems

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Continuous setting

Find $p : \Omega \to \mathbb{R}$ an approximation of the steady advection-reaction problem

$$\beta\cdot\nabla p + \mu p = s \quad \text{in } \Omega, \qquad p = p_D \quad \text{on } \partial\Omega^-.$$

- Convective vector field β
- Reaction coefficient μ
- Source terms s and Dirichlet condition p_D

Objectives

1) Accurate scheme with DoFs at vertices
2) Polyhedral/Non-conformorning 3D meshes
3) Low computational cost

Results

1) Stable scheme in the coercivity norm
2) Theoretical accuracy of order 3/2
3) Practical accuracy of order 2

Discrete scheme

Find $\mathsf{p} \in \mathcal{P}$ such that for all $\mathsf{q} \in \mathcal{P}$,

$$\mathsf{A}_{\beta,\mu}(\mathsf{p},\mathsf{p}) = \Xi(s, p_D; \mathsf{q})$$

- DoFs at vertices ($\mathcal{V}$) and cells ($\mathcal{C}$): $\mathcal{P} = \mathcal{V} \times \mathcal{C}$
- Cell-wise bilinear form: $\mathsf{A}_{\beta,\mu}(\cdot,\cdot) = \sum_{c\in\mathrm{C}} \mathsf{A}^c_{\beta,\mu}(\cdot,\cdot)$

▶ **Consistency** + **Stabilization** approach $\quad \mathsf{A}^c_{\beta,\mu}(\cdot,\cdot) = \overbrace{\mathsf{a}^c_{\beta,\mu}(\cdot,\cdot)}^{\text{Consistency}} + \overbrace{\mathsf{s}^c_{\beta}(\cdot,\cdot)}^{\text{Stabilization}}$

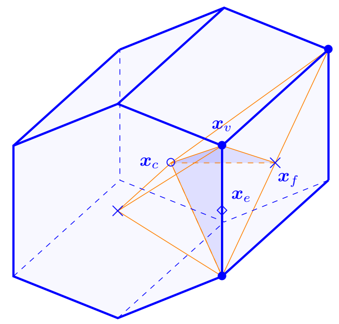
- Consistency term induced the continuous Galerkin formulation

Discrete bilinear form : $\quad \mathsf{a}^c_{\beta,\mu}(\mathsf{p},\mathsf{q}) := \int_c \beta\cdot\nabla \mathsf{L}_{\mathcal{P}_c}(\mathsf{p})\mathsf{L}_{\mathcal{P}_c}(\mathsf{q}) + \mu \mathsf{L}_{\mathcal{P}_c}(\mathsf{p})\mathsf{L}_{\mathcal{P}_c}(\mathsf{q})$

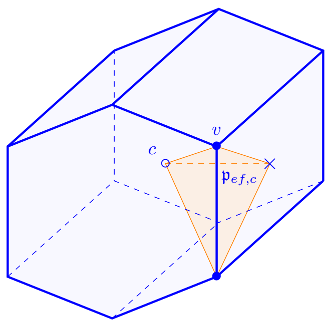
Continuous bilinear form : $\quad a^c_{\beta,\mu}(p,q) := \int_c \beta\cdot\nabla p\, q + \mu p\, q$

- Stabilization term using a local gradient jump penalty method

$$\mathsf{s}^c_{\beta}(\mathsf{p},\mathsf{q}) := h_c^2|\beta|^{-1}\int_{\mathfrak{F}_c} [\![\beta_c\cdot\nabla \mathsf{L}_{\mathcal{P}_c}(\mathsf{p})]\!][\![\beta_c\cdot\nabla \mathsf{L}_{\mathcal{P}_c}(\mathsf{q})]\!]$$



▶ **Reconstruction** map $\mathsf{L}_{\mathcal{P}_c}(\mathsf{p})(\boldsymbol{x}) = \mathsf{p}_c\ell_c(\boldsymbol{x}) + \sum_{v\in\mathrm{V}_c} \mathsf{p}_v \ell_{v,c}(\boldsymbol{x})$



$\mathbb{P}_1$-hat functions: $((\theta_v)_{v\in\mathrm{V}_c}, (\theta_f)_{f\in\mathrm{F}_c}, \theta_c)$

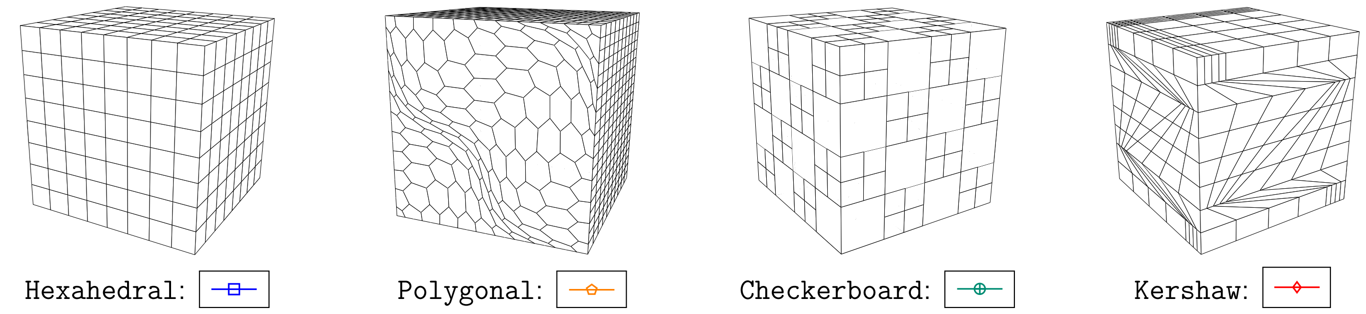
$$\forall \boldsymbol{x} \in \mathfrak{p}_{ef,c}, \quad \begin{cases} \ell_{v,c}(\boldsymbol{x}) := \delta_{ve}\theta_v(\boldsymbol{x}) + \delta_{vf}\dfrac{|f\cap \mathfrak{p}_{v,c}|}{|f|}\theta_f(\boldsymbol{x}) \\ \ell_c(\boldsymbol{x}) := \theta_c(\boldsymbol{x}) \end{cases}$$

▶ **Schur complement** w.r.t A_{cc} (diagonal). Solve a matrix of size #V instead of (#V+#C).

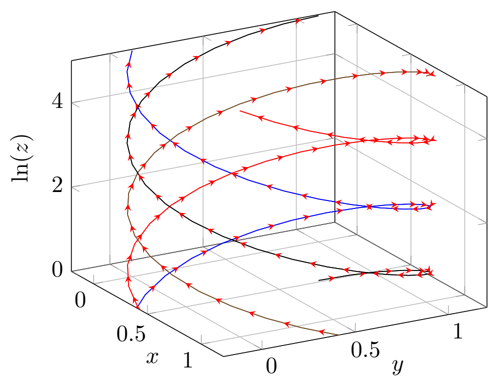
$$\begin{pmatrix} \mathsf{A}_{\mathcal{V}\mathcal{V}} & \mathsf{A}_{\mathcal{V}c} \\ \mathsf{A}_{c\mathcal{V}} & \mathsf{A}_{cc} \end{pmatrix} \mathsf{p} = \begin{pmatrix} \Xi_{\mathcal{V}} \\ \Xi_c \end{pmatrix} \xrightarrow[\text{Condensation}]{\text{Static}} \begin{pmatrix} \mathsf{A}_{\mathcal{V}\mathcal{V}} - \mathsf{A}_{\mathcal{V}c}\mathsf{A}_{cc}^{-1}\mathsf{A}_{c\mathcal{V}} & 0_{\mathcal{V}c} \\ \mathsf{A}_{cc}^{-1}\mathsf{A}_{c\mathcal{V}} & \mathsf{Id}_{cc} \end{pmatrix} \mathsf{p} = \begin{pmatrix} \Xi_{\mathcal{V}} - \mathsf{A}_{\mathcal{V}c}\mathsf{A}_{cc}^{-1}\Xi_c \\ \mathsf{A}_{cc}^{-1}\Xi_c \end{pmatrix}$$

Numerical results

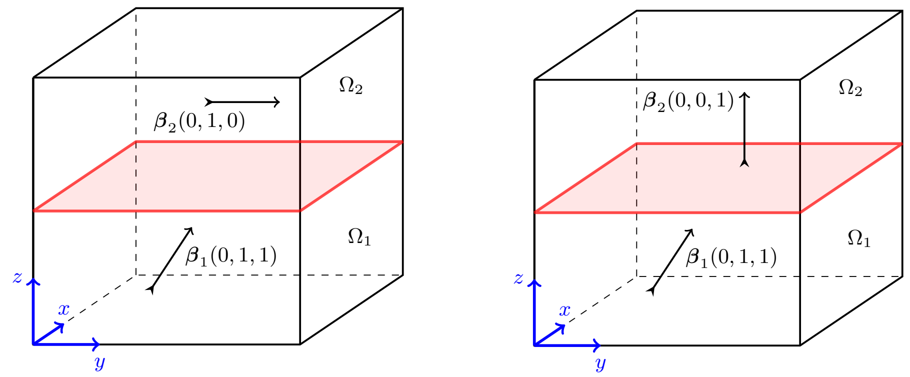
A) **Comparative study** on 3D meshes of the error $\mathbf{Er}_{\mathcal{V}}(p)$ w.r.t a smooth manufactured solution $p(x,y,z) = \sin(k_x\pi x)\sin(k_y\pi y)\sin(k_z\pi z)$

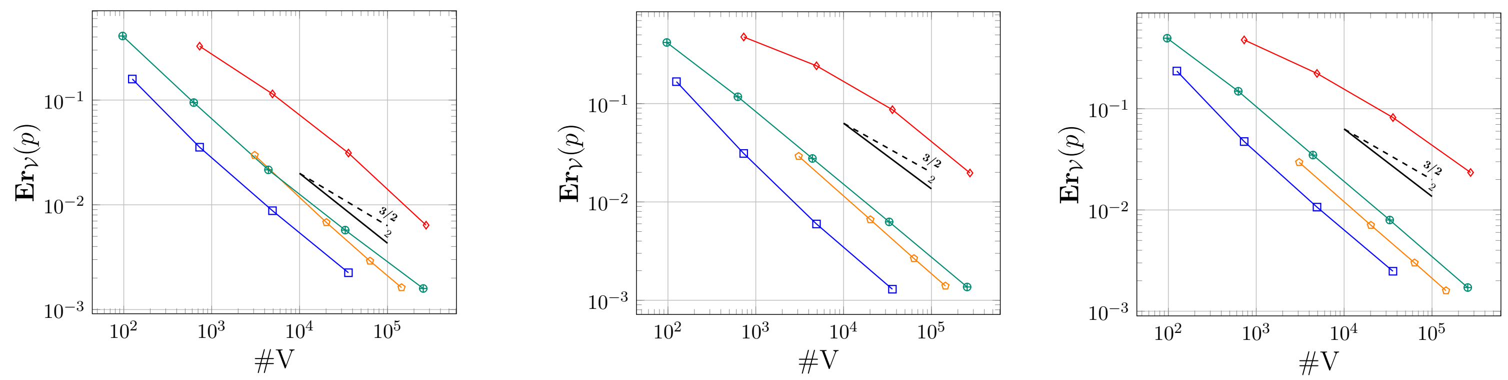


1) $\mu = 1$ and $\beta = (1/2 - y, x - 1/2, z)$

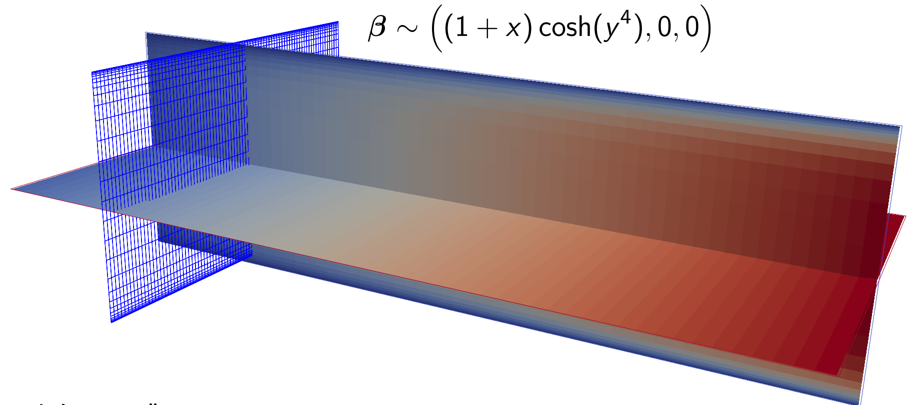


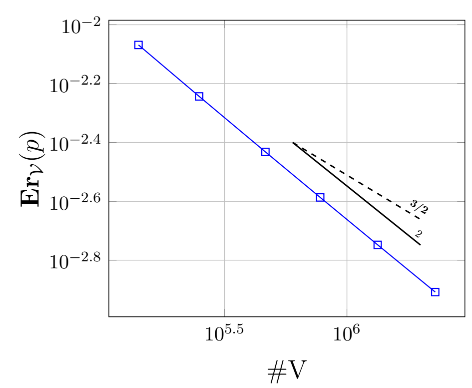
2) Smooth approximation of a piecewise constant vector field





B) **Boundary layer resolution** on adapted mesh of the convective field and the manufactured solution $p(x,y,z) \sim \sin(k_x\pi x)\sin(k_z\pi z)\cosh(y)$





Submitted paper "A vertex-based scheme on polyhedral meshes for advection-reaction problems with sub-mesh stabilization"